This document describes a solution for the voltage of a guitar pickup similar to MacDonald's, but approximating the string by a small sphere of permeable material. The volume of the sphere is chosen to be the same as MacDonald's string section of radius a_m and length w. (His a is renamed to a_m here because a is the radius of the sphere in this solution.) The potential outside a permeable sphere in a constant magnetic field is given by (http://farside.ph.utexas.edu/teaching/em/lectures/node77.html):

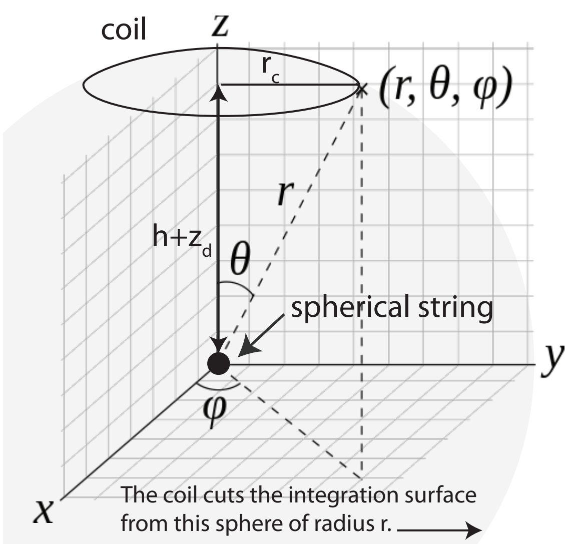
$$\phi_m = \frac{B_0}{\mu_0}\left(\frac{\mu-\mu_0}{\mu+2\mu_0}\right)\frac{a^3\cos\theta}{r^2} = B_0\mu_f a^3\frac{\cos\theta}{r^2} \tag{1}$$

where

$$\mu_f = \frac{\mu/\mu_0 - 1}{\mu/\mu_0 + 2} \tag{2}$$

is essentially one for large μ. This solution is in spherical coordinates, as used by physicists (Wikipedia: Spherical coordinate system). The sphere is located at the origin, and the center of the circular coil is located at $r = z = h$.

Rather than let the string vibrate, we hold the string still and move the coil along the z axis; perpendicular motion is not considered here. The position of the coil is $z(t) = h + z_d(t)$, where $z_d(t)$, is, for example a sinusoidal motion with some appropriate amplitude. The radius of the coil is r_c. The angle θ_c is defined by the z axis and any line segment drawn from the origin to a point on the circle defined by the coil. The length of that line segment is r. With the system in its initial state with $z_d = 0$, the angle is θ_{c0}, and r is r_0. $r_c = \tan\theta_{c0}$, constant with time, and $r = ((h + z_d(t))^2 + r_c^2)^{.5}$, variable in time.



Now we need to find the flux through the coil and then its change with time. We need to integrate over a surface bounded by the coil, but the surface need not be flat, and it is convenient to choose the surface which is a section of the sphere centered on the origin and intersecting the coil. The flux through this surface points along the r coordinate, and this coordinate has constant value on this surface. Thus the integration is easy, requiring only B_r since it is perpendicular to the surface.

B_r is found from the gradient in spherical coordinates (Wikipedia: Del in cylindrical and spherical coordinates).

$$B_r = -2B_0\mu_f a^3 \frac{\cos\theta}{r^3} \tag{3}$$

We have

$$\phi_f = -2\int_0^{\theta_s} \frac{B_0\mu_f a^3\cos\theta}{r^3} 4\pi r^2 \sin\theta d\theta = -4\pi B_0\mu_f a^3 \left.\frac{\sin^2\theta}{r}\right|_{\theta=0}^{\theta=\theta_c} = 4\pi B_0\mu_f a^3 \frac{\sin^2\theta_c}{r} \tag{4}$$

where we are integrating over 'differential rings' of area $4\pi r^2 \sin\theta$. The result is proportional to area and so we must have the angle to the second power, and the field falls of as r^3 and so the flux must fall off as r.

In order to evaulate the time derivative, we find the sine

$$\sin\theta_c = \frac{r_c}{r(t)} = \frac{r_c}{(h + z_d(t))^2 + r_c^2)^{.5}}. \tag{5}$$

Then we have, letting $C = 4\pi B_0\mu_f a^3$,

$$\frac{\partial\phi_f}{\partial t} = C\frac{\partial}{\partial t}\left[\frac{\sin^2\theta_c(t)}{r(t)}\right] = C\frac{\partial}{\partial t}\left[\frac{r_c^2}{(h + z_d(t))^2 + r_c^2)^{1.5}}\right] = \frac{-2Cr_c^2(h + z_d(t))}{(h + z_d(t))^2 + r_c^2)^{2.5}}\frac{\partial z_d(t)}{\partial t} \tag{6}$$

where the partial derivative of z_d is the velocity.

The last step is to modify C so that it contains a_m and w from MacDonald. The volume of the string segment is $2\pi a_m^2 w$. We have

$$2\pi a_m^2 w = \frac{4}{3}\pi a^3$$

which leads to

$$a^3 = \frac{3}{2}a_m^2 w$$

and

$$C = 6\pi B_0\mu_f a_m^2 w \tag{7}$$