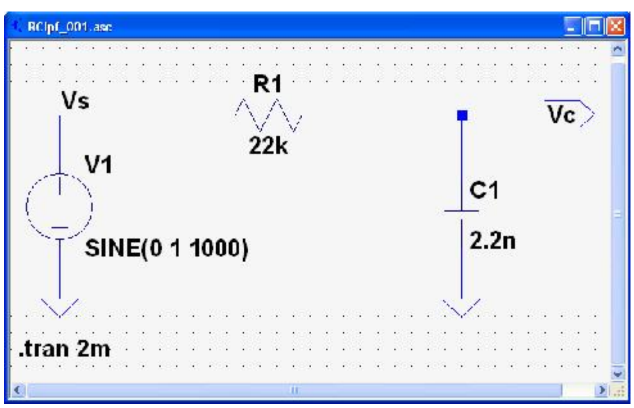
Frequency Response of RC Circuits

Peter Mathys
ECEN 1400

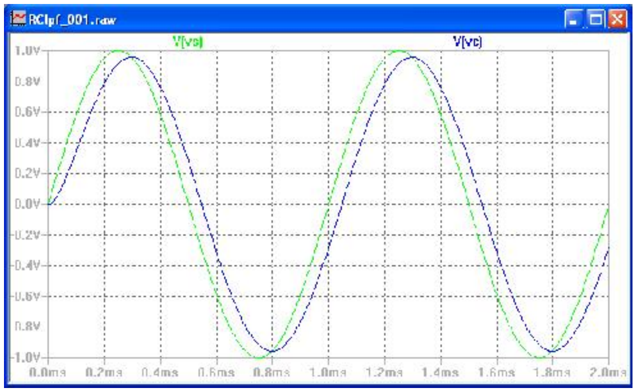
RC Circuit 1



Vs is source voltage (sine, 1000 Hz, amplitude 1 V).
Vc is voltage across capacitor.

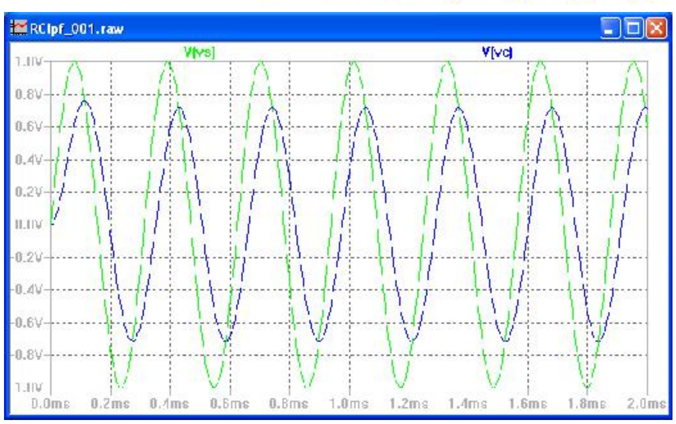
$$|X_C| = \frac{1}{2\pi f C} = \frac{1}{1.382 \times 10^{-5}} = 72.34\ \text{k}\Omega$$

Input and Output Signals: f=1kHz



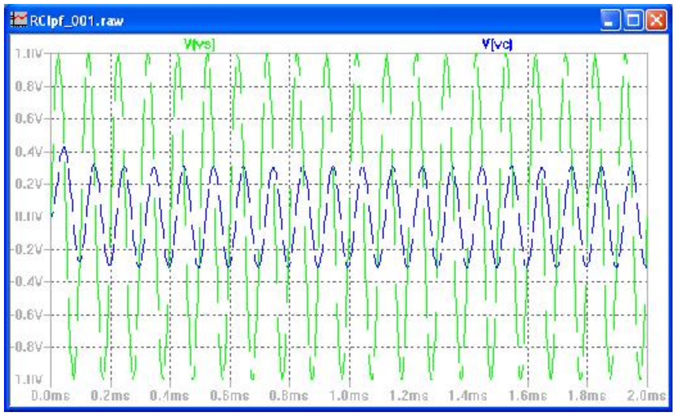
Output signal (blue) has almost same amplitude as input signal (green).

Input and Output Signals f=3.2 kHz



Now output signal (blue) is only about 0.7 times the input signal (green).
In this case X_C = 22.6 kΩ and R1 = 22 kΩ

Input and Output Signals f=10 kHz



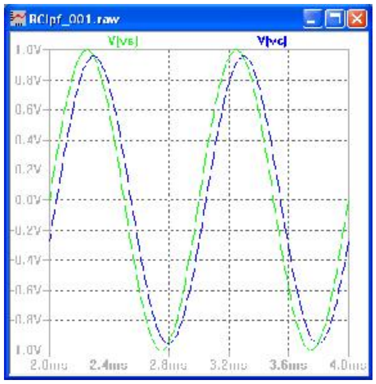
Output signal (blue) is now about 0.3 times the input signal (green).
In this case X_C = 7.23 kΩ, R1 = 22 kΩ

Thus, RC Circuit 1 passes low frequencies and attenuates high frequencies. The circuit is called a RC LPF (lowpass filter).

Sinusoidal Parameters

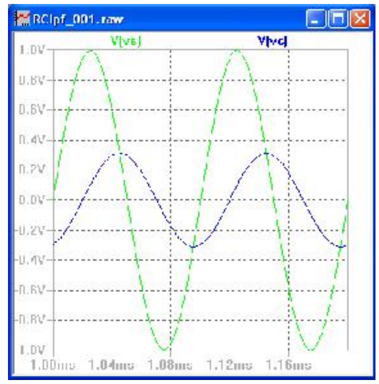
- A sinusoidal waveform
  $s(t) = A\cdot\cos(2\pi ft + \theta)$
  is characterized by its amplitude A, its frequency f and its phase θ.
- A delayed (by t_d) sinusoidal waveform is
  $s(t) = A\cdot\cos(2\pi f(t - t_d))$
- Thus $\theta = -2\pi ft_d$ (in radians) or $t_d = -\theta/(2\pi f)$ (in seconds), i.e., a delayed sinusoid corresponds to a phase shift of the sinusoid.

Zoom-In for Phase Change



1 kHz: Phase shift θ = -17⁰

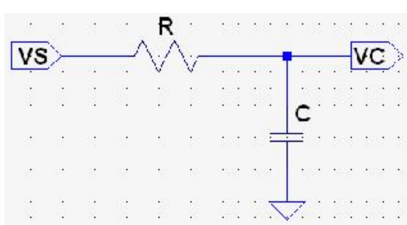
At 1 kHz $v_C(t)$ is almost equal to $v_S(t)$ in amplitude and phase.



10 kHz: Phase shift θ = -72⁰

$v_C(t)$ is delayed by about 0.02 ms (1/5 of period of 0.1 ms) from $v_S(t)$.

Analysis Using Phasors



$\underline{V}_S$ (VS) is input phasor
$\underline{V}_C$ (VC) is output phasor
$Z_C = 1/(j2\pi fC)$ is capacitor impedance

Using voltage division:

$$\underline{V}_C = \frac{\frac{1}{j2\pi fC}}{R+\frac{1}{j2\pi fC}}\underline{V}_S = \frac{1}{1+j2\pi fRC}\underline{V}_S$$

Define: $$H(f) = \frac{\underline{V}_C}{\underline{V}_S} = \frac{1}{1+j2\pi fRC}$$ System Function

Complex Number z

$z = a + j\,b$ Cartesian representation

$z = r\,e^{j\theta}$ Polar representation

Using Euler's formula: $j = \sqrt{-1}$

$z = r\,e^{j\theta} = r\cos\theta + j\,r\sin\theta$

$$\Rightarrow \boxed{\begin{array}{l} a = r\cos\theta, \quad b = r\sin\theta \\ r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right) \end{array}}$$

Conversion between Cartesian (a,b) and polar (r,θ) format

a = Re{z}: Real part of z, b = Im{z}: Imaginary part of z
r = |z|: Magnitude of z, θ = arg(z): Angle or Phase of z

System Function H(f)

- The system function H(f) is a complex-valued function of frequency f.
- For a given system, H(f) is obtained by measuring the system output $\underline{V}_O$ in response to a sinusoidal input $\underline{V}_S$ and then taking the ratio H(f)=$\underline{V}_O/\underline{V}_S$ where $\underline{V}_O$ and $\underline{V}_S$ are phasors.
- In polar representation, |H(f)| is called the **magnitude of the frequency response**.
- Also in polar representation, arg(H(f)) is called the **phase of the frequency response**.

RC LPF Frequency Response

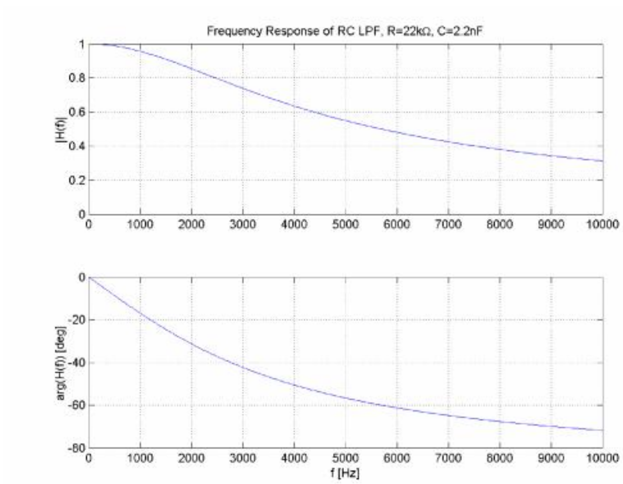
For the RC LPF:

$$H(f) = \frac{1}{1+j2\pi fRC} = \frac{1}{r\,e^{j\theta}} = \frac{1}{r}\,e^{-j\theta}$$

with $r = \sqrt{1+(2\pi fRC)^2}, \quad \theta = \tan^{-1}(2\pi fRC)$

$$\Rightarrow \begin{array}{l} |H(f)| = \frac{1}{\sqrt{1+(2\pi fRC)^2}} \\ \arg(H(f)) = -\tan^{-1}(2\pi fRC) \end{array}$$

Plot of Frequency Response



Magnitude |H(f)| of RC LPF frequency response.

Phase arg(H(f)) of RC LPF frequency response in degrees.

Matlab Script for Plots

```
%RClpf_001    Frequency Response of RC LPF

R = 22e3;                          %Resistor
C = 2.2e-9;                        %Capacitor
f1 = 0;, f2 = 10e3;, N = 500;

ff = linspace(f1,f2,N);            %Frequency axis
Hf = 1./(1+j*2*pi*ff*R*C);        %H(f)

subplot(211)
plot(ff,abs(Hf)),grid              %Magnitude of H(f)
ylim([0 1])
ylabel('|H(f)|')
title('Frequency Response of RC LPF, R=22k\Omega, C=2.2nF')
subplot(212)
plot(ff,(180/pi)*angle(Hf)),grid   %Phase of H(f)
xlabel('f [Hz]'),ylabel('arg(H(f)) [deg]')
```

Cutoff Frequency f_L

- Note that

$$|H(f)| = \frac{1}{\sqrt{1+(2\pi fRC)^2}} \approx \begin{cases} 1\,, & f < \frac{1}{2\pi RC}\,, \\ \frac{1}{2\pi fRC}\,, & f > \frac{1}{2\pi RC}\,. \end{cases}$$

- And

$$|H(f_L)| = \frac{1}{\sqrt{2}} \approx 0.707\,, \quad \text{where} \quad f_L = \frac{1}{2\pi RC}$$

- The frequency f_L at which $2\pi f_L RC=1$ is called the **cutoff frequency** of the lowpass filter.

-3 dB Frequency f_L

- |H(f)| is often expressed in decibels as

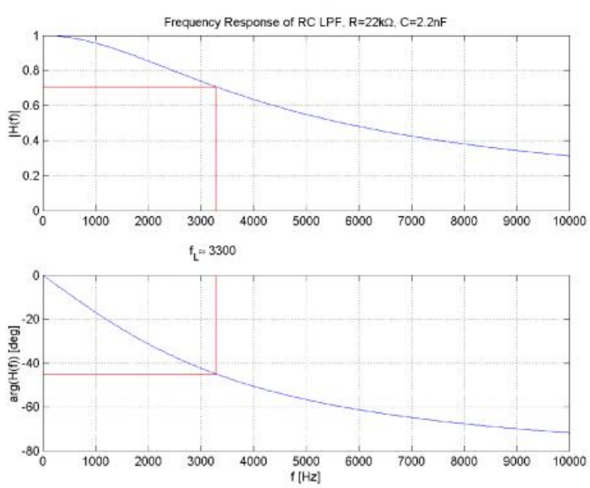
$$20\ \log_{10}(|H(f)|)\ \ \text{dB}$$

- At the cutoff frequency fL

$$20\ \log_{10}(|H(f_L)|) = -20\ \log_{10}(\sqrt{2}) = -10\ \log_{10}(2) \approx -3\,\text{dB}$$

- Thus fL is also called the **-3 dB frequency** of the lowpass filter.

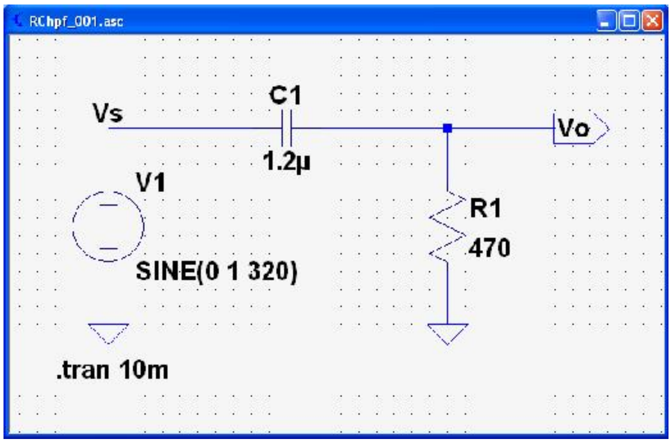
RC LPF Frequency Response with -3 dB Frequency Shown



$|H(f_L)| = 0.707$

$\arg(H(f_L)) = -45$ deg

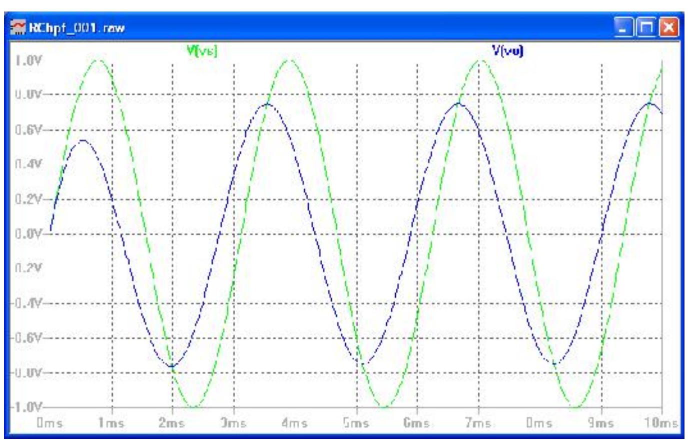
RC Highpass Filter (HPF)



For a HPF the roles of the capacitor and the resistor are interchanged.

Here the input is a sine with frequency 320 Hz and the output is taken across the 470 Ω resistor

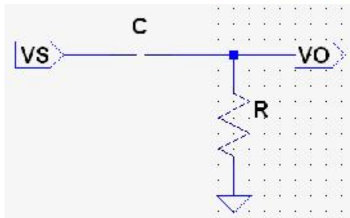
Input and Output Signals: f =320 Hz



At 320 Hz, |XC|=414.5 Ω, |H(f)|=0.75, and arg(H(f))=+41 deg

Note that for the HPF the output voltage leads compared to the input voltage. For a LPF the opposite is true, i.e., the output voltage lags compared to the input voltage.

HPF Analysis Using Phasors



$\underline{V}_S$ (VS) is input phasor
$\underline{V}_O$ (VO) is output phasor
$Z_C = 1/(j2\pi fC)$ is capacitor impedance

Using voltage division:

$$\underline{V}_O = \frac{R}{\frac{1}{j2\pi fC}+R}\underline{V}_S = \frac{j2\pi fRC}{1+j2\pi fRC}\underline{V}_S$$

System Function: $$H(f) = \frac{\underline{V}_O}{\underline{V}_S} = \frac{j2\pi fRC}{1+j2\pi fRC}$$

RC HPF Frequency Response

$$|H(f)| = \frac{2\pi fRC}{\sqrt{1+(2\pi fRC)^2}}$$ Magnitude of $H(f)$

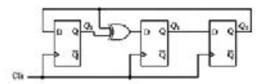
$$\arg(H(f)) = \frac{\pi}{2} - \tan^{-1}(2\pi fRC)$$ Phase of $H(f)$

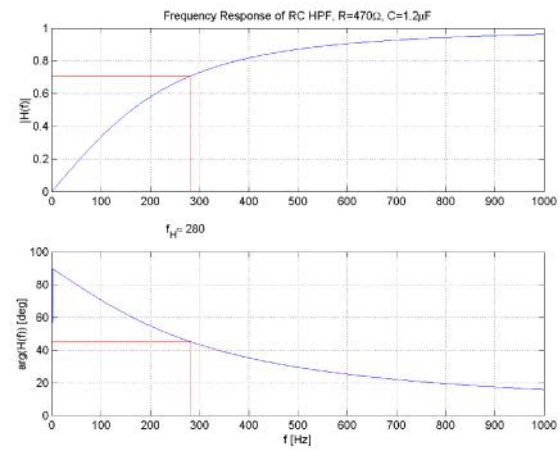
$$|H(f)| = \frac{2\pi fRC}{\sqrt{1+(2\pi fRC)^2}} \approx \begin{cases} 2\pi fRC\,, & f < \frac{1}{2\pi RC}\,, \\ 1\,, & f > \frac{1}{2\pi RC}\,. \end{cases}$$

Cutoff frequency f_H:

$$|H(f_H)| = \frac{1}{\sqrt{2}} \approx 0.707 \quad \text{for } f_H = \frac{1}{2\pi RC}$$

RC HPF Frequency Response with -3 dB Frequency Shown





$|H(f_H)| = 0.707$

$\arg(H(f_H)) = +45$ deg