Bored sat at home in isolation due to Covid-19, I decided to test my brain working out guitar bridge placement, and intonation

If the distance from the nut to the 12th fret is 315mm then the centreline of the bridge saddles should be 630mm? Wrong. What you have missed is the fact that when you depress the string at the 12th fret, you must stretch the string slightly; if the string has stretched, then its tuning must rise slightly, just like when you bend a string.

Since the positions of the each fret is fixed by the relationship between each note; (the ratio of the 12th root of 2 = 1.059463094), so in order to correct this slight error, the only thing we can alter is the relative position of the bridge; hence the term bridge compensation.

Figure1 shows an imaginary guitar with a string length of 630mm. I am not a guitarist, so I have no real idea of string to fretboard clearance, and so I have taken an exaggerated string clearance above the fretboard of 5mm, which will give a nice big bridge compensation figure. Unfretted, the string length is 630mm, but when you depress the string at the 12th fret, it must elongate slightly. We can work out the stretched length using Pythagoras' Theorem (for right angled triangles).
Pythagoras tells us that the hypotenuse (h) will be

$$h=\sqrt{315^2+5^2} = \sqrt{99250}$$

which gives h = as 315.03968mm.

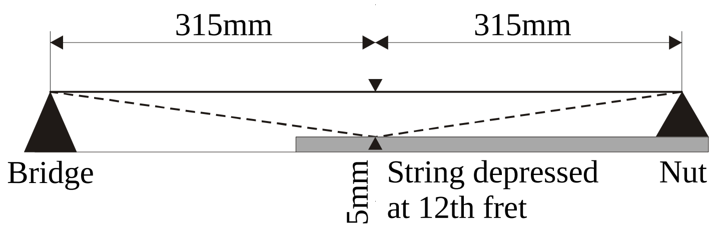

Figure 1.

Remember, this is only one half of the string, so for the overall length, we need twice this, which is 630.079mm. The increase in string length therefore comes to 0.079mm.

Because the string has been stretched slightly, it has more tension, so its pitch has gone up slightly. This then is why we need bridge compensation; we need to adjust the distance between the 12th fret and the bridge to correct for this slight rise in pitch.

You can find the tension of a tuned string from any string manufacturer's website, but if you are bored, like me, I worked it out from scratch. For this example, I am taking an E string of 0.28mm diameter, tuned to 326Hz. Now comes the physics, tension in a stretched string is given by Taylor's formula. [1]

$p = 4l^2n^2a\rho$.

Where p is the tension on the string in Newton's
l is the string length in metres, = 0.63m

n is the pitch = 326Hz
a is the cross sectional area of the string in square metres = $\pi r^2 = \pi \times 0.00014^2 = 6.157 \times 10^{-8} m^2$
ρ is the density of the string material = $7850kgm^3$ [2]

So, we have

$p = 4 \times 0.63^2 \times 326^2 \times 6.157 \times 10^{-8} \times 7850 = 81.558N$ for the open string.

Now, a good mathematician might be able to see a way to evaluate the resultant force on the string when it is stretched by depressing at the12th fret, but I cannot see a direct solution, so I had to do the evaluation this in an indirect way.

Young's Modulus of Elasticity (E) for this guitar string is $206kNmm^2$. This means that if we applied a force of 206kN to a $1mm^2$ cross sectional area string, its length would double. The string in this example is 0.28mm diameter, and so has a cross sectional area of $0.061575mm^2$. In order to double the length of this string, the force then would be 206 x 0.061575, or 12.68445kN. In the example, the length is not doubled, but increased by only 0.078mm, which is 630 / 0.079 giving a ratio of 1:7975. Working out the force change on the string from this ratio, 12684.45/7975 = 1.5905N.

From this roundabout route, the resultant tension on the string, when depressed at the 12^{th} fret is 81.55524 + 1.57045N = 83.12569N.

Now we can use Taylor's formulae again, but now transposing it to make the string length the subject.

$$l=\sqrt{\frac{p}{4n^2 a\rho}}$$

$$l=\sqrt{\frac{83.12569}{4\times 326^2 \times 6.157\times 10^{-8}\times 7850}} = \sqrt{0.404577066} = 0.63606m$$

The bridge, for this one string, must therefore lie 636mm from the nut, or 321mm from the 12^{th} fret, if the octave note is to have the correct pitch when the string is stopped at the 12^{th} fret. Note that this only holds for plain strings, wound strings are rather more complex, so I will leave this for a later article. It should also be noted that I have taken data for a nickel steel string, for a different material, Young's modulus and the density will change.

References.
[1] = Schaum's Physics for Engineering and Science.
[2] = Kemp's Engineers Year Book (any edition).